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Many-particle systems
VIII. Saturation and a lower-bound shell model

G M Horton
Department of Physics, Chelsea College, University of London, London SW6, UK

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Abstract. The lower-bound shell model developed by Carr and Post is applied to a certain class of non-local two-body interactions and shows saturation. The case of the Yamaguchi interaction is worked out in detail.

1. Introduction

In this paper we prove that the lower-bound shell model established in paper VI (Carr and Post 1968) shows saturation for a certain class of non-local two-body interactions. These interactions are constructed in such a way that the two-body system has at most a finite number of bound states.

A specific instance of a non-local two-body interaction is worked out in detail. The two-body interaction is non-local, factorizable, and acts in S states only. In order to use the lower-bound shell model the total interaction must be expressible as a sum of two-body terms and is not factorizable itself. This particular example was used to discuss the deuteron by Yamaguchi (1954) and the three-nucleon problem by Mitra (1962, 1963).

2. Formulation of the problem

The N particle hamiltonian of the exact problem is

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{P}_i^2}{2m} + \sum_{i<j=1}^{N}\sum V_{ij}$$

where m is the mass of each particle and the ith particle has momentum vector $\boldsymbol{P}_i$. The two-body interaction for particles i and j, V_{ij}, is translation invariant and spin independent. The last restriction is not essential but simplifies the discussion. In the momentum representation V_{ij} is defined by

$$V_{ij}\psi(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_N) = \int V(\boldsymbol{P}_i - \boldsymbol{P}_j; \boldsymbol{P}'_i - \boldsymbol{P}'_j) \prod_{k \neq i,j} \delta(\boldsymbol{P}_k - \boldsymbol{P}'_k)\psi(\boldsymbol{P}'_1, \ldots, \boldsymbol{P}'_N)\, d\tau'.$$

$\psi(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_N)$ is a translation invariant function of $3(N-1)$ relative momenta and $d\tau'$ is the volume element of the $3(N-1)$ dimensional momentum space. The function V is the same for all pairs of particles and the product of the delta functions ensures that there is only interaction between particles i and j. Since the derivation of the lower-bound shell model involves letting the mass of particle 1 tend to infinity, it is important to note that there is no explicit dependence on the mass of each particle in the definition of V_{ij}.

The derivation of the lower-bound shell model in paper VI remains valid in the momentum representation and, since in our H the only explicit dependence on the mass of each particle is in the kinetic energy terms, we get a shell-model hamiltonian $\mathscr{H}$

$$\mathscr{H} = \sum_{i=2}^{N} h_i$$

$$h_i = \frac{\boldsymbol{P}_i^2}{2m} + \frac{N}{2} V_{1i}.$$

In the lower bound shell model particle one is fixed and the other $(N-1)$ particles interact only with particle one by the one-body interaction V_{1j}. The lower bound to the ground state energy E_0 of the original N particle problem is given, in the case of fermions, by putting a particle in each state of the shell model. If ϵ_i are the energies of the states of particles moving in the field of particle one then

$$|E_0| \leqslant \sum_{i=1}^{N-1} |\epsilon_i|.$$

The sum is over the lowest $(N-1)$ states.

If the ϵ_i are such that $\lim_{N\to\infty} |\epsilon_i|/N$ is non-infinite, and if the number of bound states of the one-body problem does not exceed some fixed integer (n, say) then $\lim_{N\to\infty} |E_0|/N$ is non-infinite.

On the other hand, the $\lim_{N\to\infty} |E_0|/N$ can be assumed to be nonzero since E_0 will be bounded from above by the energy of $\frac{1}{2}N$ independent pairs of particles and we may suppose the two-body problem to be bound.

In the next section we consider one-body potentials that have the desired properties. Thus, we will have shown that the lower-bound shell model shows saturation in the sense that $\lim_{N\to\infty} |E_0|/N$ is finite.

3. One-body problems with at most *n* bound states

The one-particle Schrödinger wave equation in the momentum representation is

$$\frac{\boldsymbol{P}^2}{2m}\psi(\boldsymbol{P}) + \int V(\boldsymbol{P};\boldsymbol{P}')\,\mathrm{d}\boldsymbol{P}'\psi(\boldsymbol{P}') = E\psi(\boldsymbol{P}).$$

The interaction must satisfy the usual invariance requirements and be hermitian

$$V(\boldsymbol{P};\boldsymbol{P}') = V^*(\boldsymbol{P}';\boldsymbol{P}).$$

We now consider those functions $V(\boldsymbol{P};\boldsymbol{P}')$ that can be written in the form

$$V(\boldsymbol{P};\boldsymbol{P}') = \sum_{i=1}^{n} \lambda_i f_i(\boldsymbol{P}) g_i(\boldsymbol{P}')$$

where the λ_i are numbers and the $f_i(\boldsymbol{P})$ are n linearly independent functions. The $g_i(\boldsymbol{P}')$ are related to the $f_i(\boldsymbol{P})$ by the condition of hermiticity.

The wave equation is now

$$\frac{\boldsymbol{P}^2}{2m}\psi(\boldsymbol{P})+\sum_{i=1}^{n}\lambda_i a_i f_i(\boldsymbol{P}) = E\psi(\boldsymbol{P})$$

where

$$a_i = \int g_i(\boldsymbol{P}')\,\mathrm{d}\boldsymbol{P}'\psi(\boldsymbol{P}').$$

The solution is

$$\psi(\boldsymbol{P}) = \sum_{i=1}^{n}\lambda_i a_i \frac{f_i(\boldsymbol{P})}{E-\boldsymbol{P}^2/2m}.$$

$\psi(\boldsymbol{P})$ is a linear combination of at most n linearly-independent functions and there can therefore be at most n linearly-independent solutions of the Schrödinger wave equation. It is necessary to impose conditions on the $f_i(\boldsymbol{P})$ and $g_i(\boldsymbol{P})$ such that the integrals defining the a_i exist and $\psi(\boldsymbol{P})$ be square integrable. If we choose $V(\boldsymbol{P};\boldsymbol{P}')$ to be a square integrable, symmetric function of $\boldsymbol{P}$ and $\boldsymbol{P}'$ then it may be expanded in a doubly orthogonal series

$$V(\boldsymbol{P};\boldsymbol{P}') = \sum_{i=1}^{\infty}\lambda_i f_i(\boldsymbol{P})f_i(\boldsymbol{P}')$$

where the λ_i are real numbers and the $f_i(\boldsymbol{P})$ are members of an orthonormal set of square-integrable functions (see eg Coleman 1963). The number of terms in the sum is unique if it is finite even though the $f_i(\boldsymbol{P}')$ are not unique. The solution $\psi(\boldsymbol{P})$ of the Schrödinger wave equation corresponds to a bound state for some values of the λ_i and m.

A special case of the above is

$$V(\boldsymbol{P};\boldsymbol{P}') = -\lambda f(|\boldsymbol{P}|)f(|\boldsymbol{P}'|)$$

where λ is a real number and f is a real function. This interaction acts in S states only since $\int f(|\boldsymbol{P}'|)\psi(\boldsymbol{P}')\,\mathrm{d}\boldsymbol{P}'$ vanishes unless $\psi(\boldsymbol{P}')$ has no dependence on the orientation of $\boldsymbol{P}'$. It is also a factorizable interaction. The one-body problem has then at most one bound energy level. The solution of the Schrödinger wave equation is

$$\psi(\boldsymbol{P}) = -\frac{a\lambda f(|\boldsymbol{P}|)}{E-\boldsymbol{P}^2/2m}.$$

For a bound state E is negative and so

$$\psi(\boldsymbol{P}) = \frac{a\lambda f(|\boldsymbol{P}|)}{\boldsymbol{P}^2/2m+|E|}$$

$$a = \int f(|\boldsymbol{P}'|)\psi(\boldsymbol{P}')\,\mathrm{d}\boldsymbol{P}' = a\lambda\int\frac{f^2(|\boldsymbol{P}'|)\,\mathrm{d}\boldsymbol{P}'}{\boldsymbol{P}'^2/2m+|E|},$$

therefore

$$\frac{1}{\lambda} = \int\frac{f^2(|\boldsymbol{P}'|)\,\mathrm{d}\boldsymbol{P}'}{\boldsymbol{P}'^2/2m+|E|}.$$

The last relation can be solved to find $|E|$ in terms of λ. One can obtain $|E|$ proportional to λ, for large λ, by choosing $f(|\boldsymbol{P}|)$ suitably.

4. The Yamaguchi interaction and the lower-bound shell model

We choose for $f(|\boldsymbol{P}|)$ the form used by Yamaguchi

$$f(|\boldsymbol{P}|) = \frac{1}{\boldsymbol{P}^2+\beta^2}$$

where β is a constant.

The energy of the bound level is given by

$$\frac{1}{\lambda} = \int \frac{4\pi P^2\,\mathrm{d}P}{(P^2+\beta^2)(P^2/2m+|E|)} \qquad P = |\boldsymbol{P}|.$$

After integration we find

$$|E|\beta^2 = \left(\sqrt{(\lambda\pi^2\beta)}-\frac{\beta^2}{\sqrt{(2m)}}\right)^2.$$

In the shell model the parameter λ is changed to $\lambda N/2$ so that the only bound energy level has energy ϵ_0

$$|\epsilon_0|\beta^2 = \left(\sqrt{(\tfrac{1}{2}\lambda N\pi^2\beta)}-\frac{\beta^2}{\sqrt{(2m)}}\right)^2.$$

If the particles are fermions we may put two particles in each level or, if nucleons, we may put four particles in each level and we then have $E_0 \geqslant \gamma\epsilon_0$ with $\gamma = 2$ or 4,

$$\lim_{N\to\infty}\frac{|E_0|}{N} \leqslant \lim_{N\to\infty}\frac{\gamma|\epsilon_0|}{N} = \frac{\gamma\lambda\pi^2}{2\beta}.$$

We also have

$$\lim_{N\to\infty}\frac{|E_0|}{N} \geqslant \frac{1}{2}\left(\sqrt{(\lambda\pi^2\beta)}-\frac{\beta^2}{\sqrt{m}}\right)^2\frac{1}{\beta^2}.$$

This latter estimate is given by considering $N/2$ independent pairs of particles. Thus the shell model shows saturation.

This lower bound may be a poor approximation to the exact value for this interaction. One indication that the result will not be a good approximation for all values of λ, β and m is the fact that the lower bound for the energy per particle contains no dependence on m when we proceed to the limit $N = \infty$ whereas one would expect that as m was decreased the system would become unbound for some finite value of m independent of N†.

5. Conclusion

It has been shown that the lower-bound shell model may be used to discuss saturation for a class of non-local interactions and, in particular, that the choice of the Yamaguchi interaction leads to saturation.

† We are grateful to Dr R Huby for pointing out this latter fact.

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